ITERATIVE METHODS FOR CONSTRAINED PHASE RECOVERY

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THE SIGNAL RECOVERY PROBLEM

A signal x is to be estimated from data y and some *a priori* knowledge:

- Image restoration: $y \in \mathcal{H}$ is a blurred and noise corrupted version of x.
- Image reconstruction: y is a collection of signals related to x.

Four basic elements are required to solve this problem:

1. A data formation model, i.e., a mathematical description of the relation between the original image x and the recorded data y. For instance,
$$y = Lx + u.$$
2. Some a priori information about the original signal x, the noise sources, etc.
3. A recovery criterion defining the solutions to the problem.
4. A solution method, i.e., a numerical algorithm that will produce an image that satisfies the recovery criterion.

MATHEMATICAL SET-UP

- The original signal lies in a real Hilbert space $\mathcal{H}$ with
  - scalar product $\langle \cdot \mid \cdot \rangle$
  - norm $\| \cdot \|$
  - distance d.

- Typically, $\mathcal{H}$ is (a subspace of) $\mathcal{L}^2(\Omega, \mathcal{A}, \mu)$. In particular:
  - $\mathcal{H} = L^2(\mathbb{R}^M)$ for M-dimensional analog signals.
  - $\mathcal{H} = \ell^2(\mathbb{Z}^M)$ for M-dimensional discrete-space signals.
  - $\mathcal{H} = (\mathbb{R}^N)^M$ for discrete-space, finite-extent signals.

- The *a priori* knowledge and the data give rise to a family of constraints $(\Psi_i)_{i \in I}$ associated with the property sets

$$(\forall i \in I) \;\; S_i = \big\{ x \in \mathcal{H} \mid x \text{ satisfies } \Psi_i \big\}.$$

- The feasibility set is

$$S = \bigcap_{i \in I} S_i.$$

MATHEMATICAL SET-UP (cont'd)

- The recovery problem can be posed as a minimization problem:

$$\text{Find } x \in S \ \text{ such that } \ f(x) = \inf f(S),$$

  where $f : \mathcal{H} \to \,]-\infty, +\infty]$ is an optimality criterion.

- In many cases f cannot be determined objectively and is constant. The problem is then a feasibility problem:

$$\text{Find } x \in S.$$

MATHEMATICAL SET-UP (cont'd)

- The Fourier transform of $x \in \mathcal{H}$ is denoted by $\widehat{x}$.
  - In $L^2(\mathbb{R}^M)$,

$$\begin{aligned} \widehat{x}\colon \quad \mathbb{R}^M &\to \mathbb{C} \\ \nu &\mapsto \int_{\mathbb{R}^M} x(t)\exp(-i2\pi\nu\cdot t)dt. \end{aligned}$$

  - In $\mathcal{H} = \ell^2(\mathbb{Z}^M)$,

$$\begin{aligned} \widehat{x}\colon \quad [-1/2, 1/2[^M &\to \mathbb{C} \\ \nu &\mapsto \sum_{n\in\mathbb{Z}^M} x(n)\exp(-i2\pi\nu\cdot n). \end{aligned}$$

  - In $\mathcal{H} = (\mathbb{R}^N)^M$, we obtain the DFT

$$\begin{aligned} \widehat{x}\colon \quad \mathcal{N} &\to \mathbb{C} \\ k &\mapsto \sum_{n\in\mathcal{N}} x(n)\exp\left(-i\frac{2\pi}{N}k\cdot n\right), \end{aligned}$$

  where $\mathcal{N} = \{0,\ldots,N-1\}^M$.

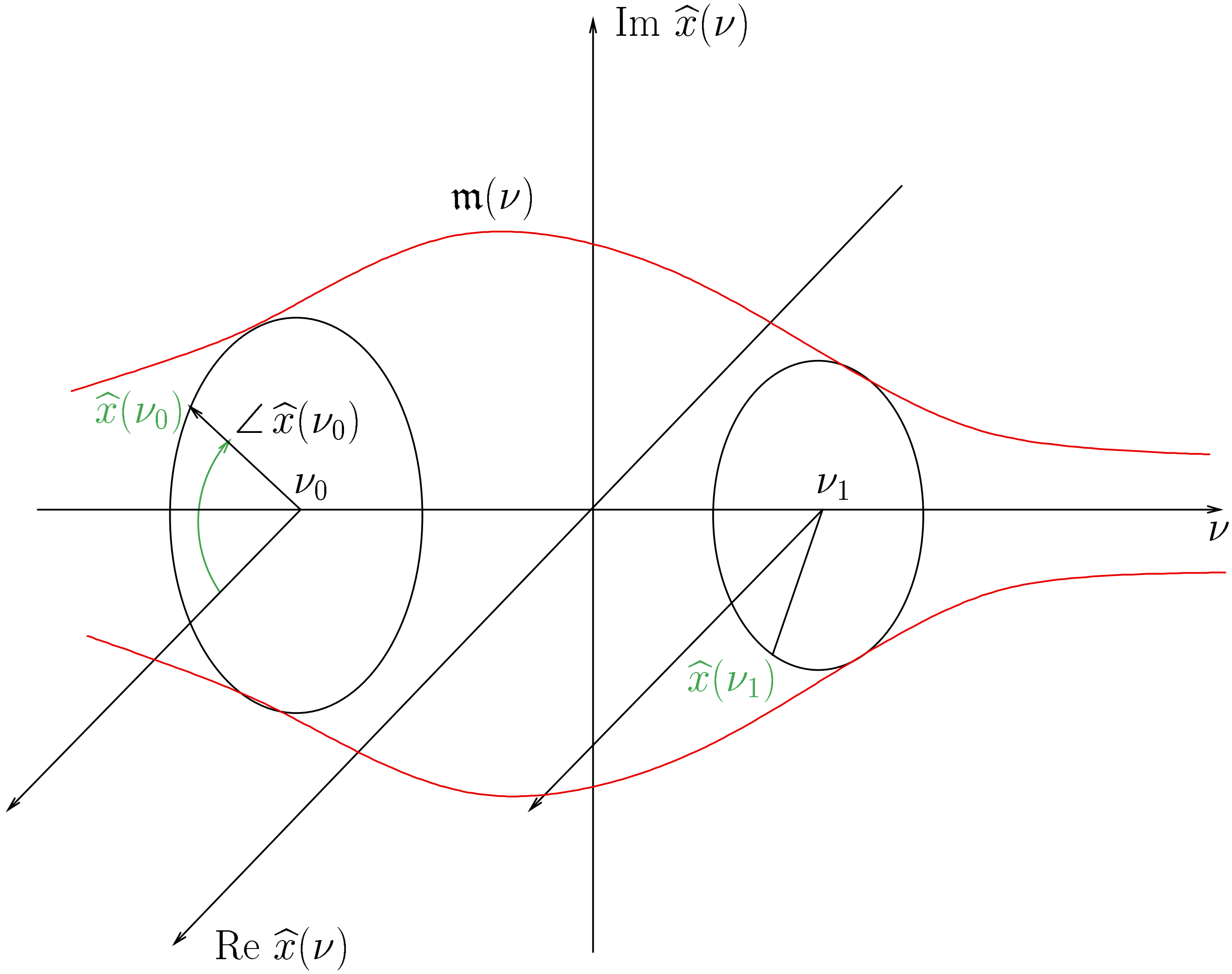
SIGNAL RECOVERY WITH FOURIER INFORMATION

Pieces of information that may be available about $\widehat{x}$ ($A \subset \mathbb{R}^M$):

- Support: $\widehat{x} 1_A = 0$ (closed vector subspace)
- Moments: $\int s\, \widehat{x} = \eta$, e.g., $s\colon \nu \mapsto |\nu|^p$
  (closed affine hyperplane)
- Positivity: $\widehat{x} 1_A \geq 0$
  (closed convex cone)
- Phase: $\widehat{x} 1_A = |\widehat{x}| \exp(i\varphi) 1_A$, φ prescribed
  (closed convex cone)
- Bounded energy: $\|\widehat{x} 1_A\|^2 \leq \eta$
  (closed ball)
- Bounded residual energy: $\|\widehat{y} - \widehat{l}\, \widehat{x}\|^2 \leq \eta$
  (closed convex set)
- Upper modulus envelope: $|\widehat{x}| 1_A \leq \mathfrak{m} 1_A$, $\mathfrak{m}$ prescribed
  (closed convex set)
- Modulus envelope : $\mathfrak{n} 1_A \leq |\widehat{x}| 1_A \leq \mathfrak{m} 1_A$, $\mathfrak{m}, \mathfrak{n}$ prescribed
  (nonconvex set, unless $\mathfrak{m} 1_A \equiv 0$)
- Modulus : $|\widehat{x}| 1_A = \mathfrak{m} 1_A$, $\mathfrak{m}$ prescribed
  (nonconvex set)

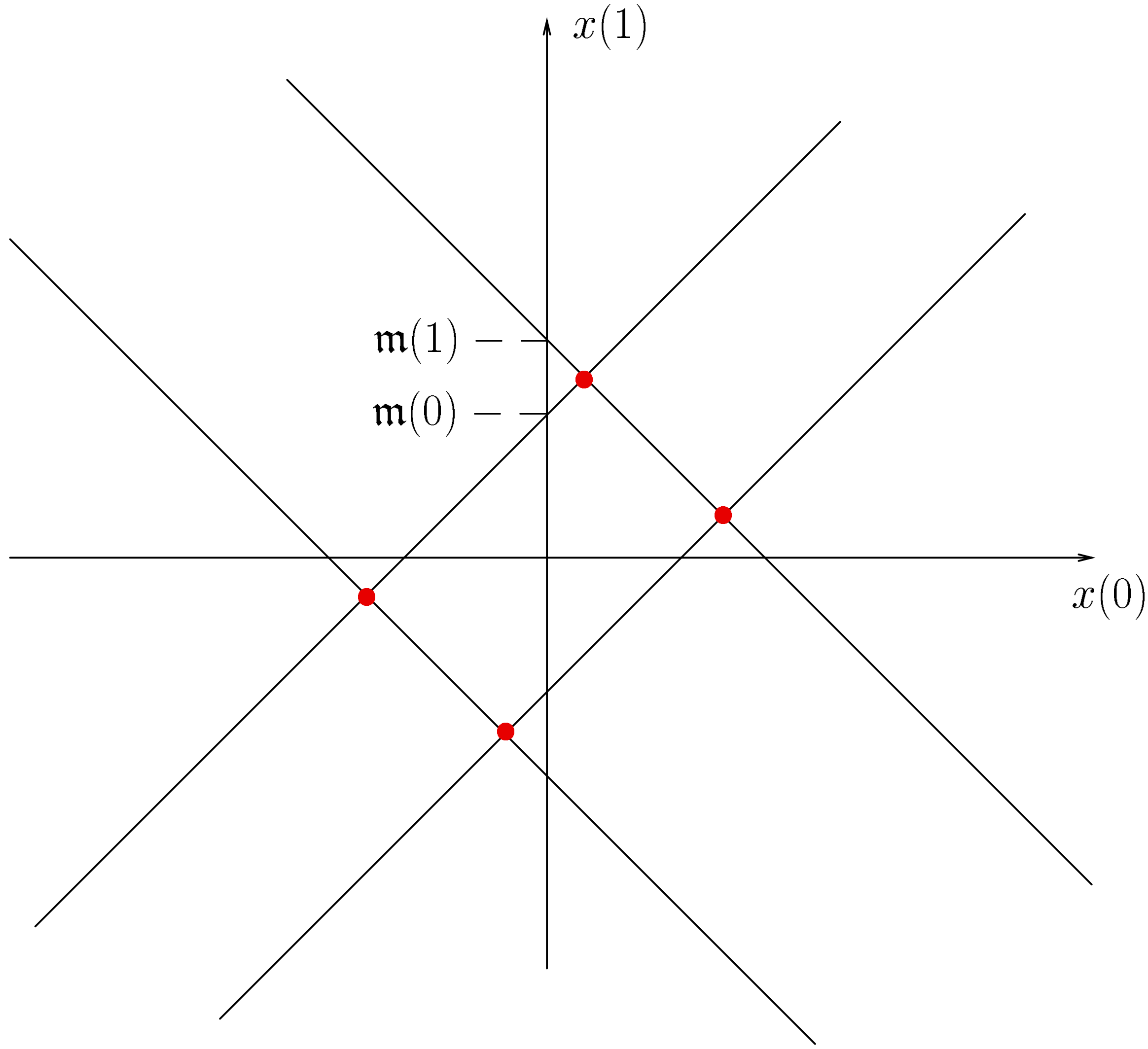
IMPORTANCE OF PHASE INFORMATION

Clockwise from upper left: x, y, $\mathcal{F}^{-1}\big(|\widehat{x}|\exp(i\angle\widehat{y})\big)$, and $\mathcal{F}^{-1}\big(|\widehat{y}|\exp(i\angle\widehat{x})\big)$.



1-D analog representation of the constraint $|\widehat{x}| = \mathfrak{m}$.

NONCONVEXITY OF THE MODULUS CONSTRAINT II



2-point DFT representation of the constraint $|\widehat{x}| = \mathfrak{m}$.

$$\begin{cases} |x(0)+x(1)| = |\widehat{x}(0)| = \mathfrak{m}(0) \\ |x(0)-x(1)| = |\widehat{x}(1)| = \mathfrak{m}(1). \end{cases}$$

The feasibility set consists of 4 isolated points.

CONVEXIFICATION OF THE MODULUS CONSTRAINT

- A number of efficient algorithms are available to solve convex minimization and feasibility problems.

- Can the phase retrieval problem be convexified in order to take advantage of these tools?

- We describe three convexification approaches.

CONVEXITY OF THE MODULUS CONSTRAINT I

- Instead of working with x itself, work with its autocorrelation $r_{xx} = x \star \overline{x}^{\vee}$.
- The spectral density of x is $S_{xx} = \widehat{r_{xx}} = |\widehat{x}|^2$.
- In the autocorrelation space, the nonconvex modulus constraint

$$|\widehat{x}|1_A = \mathfrak{m}1_A$$

therefore becomes a convex (actually affine) constraint

$$\widehat{r_{xx}}1_A = \mathfrak{m}^2 1_A$$

and its carries implicitly the conical convex constraint

$$\widehat{r_{xx}} \geq 0.$$

- Conceptually, a solution can be obtained as follows:
  - If other relevant constraints on the signal yield convex constraints in the autocorrelation space, solve the resulting convex problem and obtain r_{xx}.
  - Use spectral factorization to construct $\widehat{x}$ from $\widehat{r_{xx}}$ (possible only in certain 1-D problems in general).

CONVEXITY OF THE MODULUS CONSTRAINT II

- A subset S_i of a real vector space $(E, \boxplus, \boxdot)$ is convex if

  $$(\forall \alpha \in]0,1[)(\forall (x,y) \in S_i^2) \ \ (\alpha \boxdot x) \boxplus ((1-\alpha) \boxdot y) \in S_i.$$

- Let ℓ be the subset of ℓ^1 of discrete-time 1-D signals whose Fourier transform is nonzero a.e.

- Define

  $$(\forall \alpha \in \mathbb{R})(\forall (x,y) \in \ell \times \ell) \ \ \begin{cases} x \boxplus y = x \star y \\ \alpha \boxdot x = \mathcal{F}^{-1}\big(\exp(\alpha \ln(\widehat{x}))\big). \end{cases}$$

- Fact: $(\ell, \boxplus, \boxdot)$ is a vector space.

- It was observed by Çetin that the modulus constraint set

  $$S_i = \big\{x \in \ell \mid |\widehat{x}| 1_A = \mathfrak{m} 1_A\big\}$$

  is convex in $(\ell, \boxplus, \boxdot)$.

CONVEXITY OF THE MODULUS CONSTRAINT II (cont'd)

- The scalar product between two signals x and y in ℓ can be defined as

$$\langle x \mid y \rangle = \int_{-1/2}^{1/2} \ln(\widehat{x}(\nu))\overline{\ln(\widehat{y}(\nu))}\, d\nu.$$

- In some problems, other useful constraints may be convex in ℓ and one can therefore solve the phase retrieval problem in a convex optimization framework.

- Unfortunately, many useful constraints which are convex in ℓ^2 are no longer convex in ℓ.

CONVEXITY OF THE SUBMODULUS CONSTRAINT III

Q: How much is lost by replacing the nonconvex set

$$S_i = \left\{ x \in \mathcal{H} \mid |\widehat{x}| 1_A = \mathfrak{m} 1_A \right\}$$

by its convex hull, i.e., the convex set

$$S_i = \left\{ x \in \mathcal{H} \mid |\widehat{x}| 1_A \leq \mathfrak{m} 1_A \right\}?$$

CONVEXITY OF THE SUBMODULUS CONSTRAINT III (cont'd)

A: No general answer but:

- If the only nonconvex constraint is the modulus constraint, an approximate solution should be sought via a convexity algorithm with the submodulus set.
- By projecting this solution onto the exact modulus set, one can measure a "feasibility gap" and assess whether further processing is necessary.
- The above convexification approach is quite common in signal feasibility problems. For instance, with the linear model

$$y = Lx + b$$

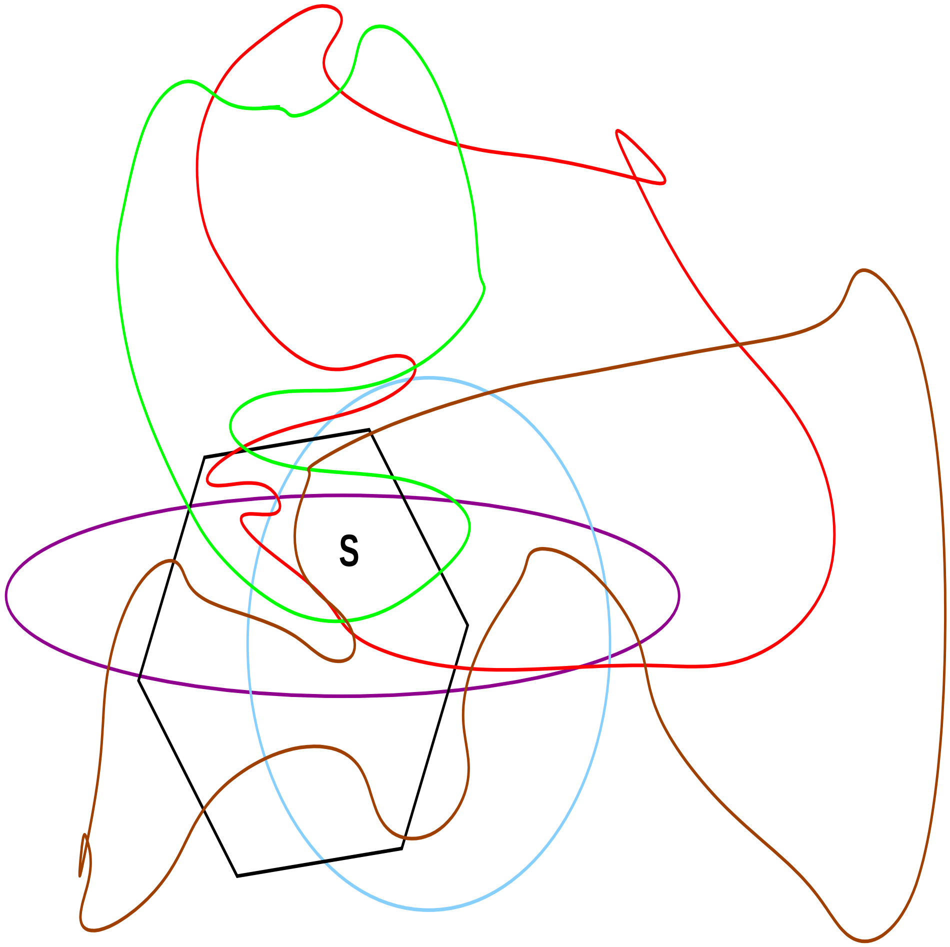
the nonconvex exact residual variance set

$$S_v = \left\{x \in \mathcal{H} \mid \eta_- \leq \|Lx - y\|^2 \leq \eta_+\right\}$$

is replaced by its convex hull

$$S_v^+ = \left\{x \in \mathcal{H} \mid \|Lx - y\|^2 \leq \eta_+\right\}$$

- This approach gives satisfactory results if the remaining constraints are discriminating enough.



Find $x \in S$.

FEASIBILITY IN THE PRODUCT SPACE

- Suppose we have m constraints and define the m-fold cartesian product space $\mathbf{H} = \mathcal{H}^m$
- Take weights $(\omega_i)_{1 \le i \le m}$ in $]0,1]$ such that $\sum_{i=1}^m \omega_i = 1$.
- $\mathbf{H}$ is a Hilbert space with norm

$$|||\cdot|||\colon (x_1, \ldots, x_m) \mapsto \sqrt{\sum_{i=1}^{m} \omega_i \|x_i\|^2}.$$

- In $\mathbf{H}$, define the cartesian product of the constraint sets

$$\mathbf{S} = S_1 \times \cdots \times S_m$$

and the diagonal subspace

$$\mathbf{D} = \big\{(x, \ldots, x) \mid x \in \mathcal{H}\big\}.$$

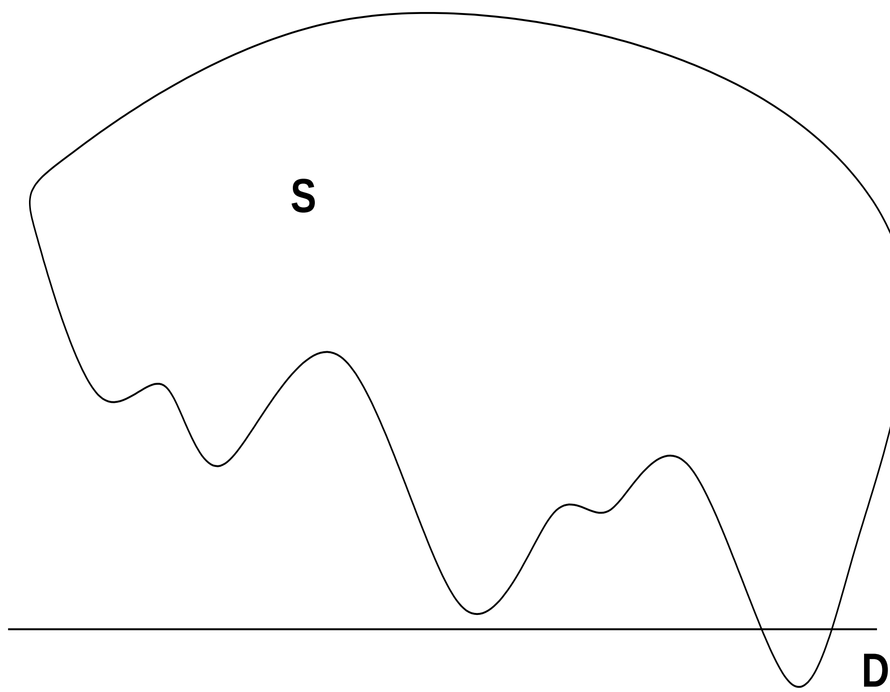
FEASIBILITY IN THE PRODUCT SPACE (cont'd)

- The m-set feasibility problem in $\mathcal{H}$

$$\text{Find } x \in S = \bigcap_{i=1}^{m} S_i$$

is equivalent to the 2-set feasibility problem in $\mathbf{H}$

$$\text{Find } \mathbf{x} \in \mathbf{S} \cap \mathbf{D}.$$



METRIC PROJECTIONS

- Let S_i be a closed set

- The projector onto S_i is the set-valued mapping

$$\Pi_i \colon x \mapsto \left\{ p \in S_i \mid \|x - p\| = \inf_{y \in S_i} \|x - y\| \right\}$$

- The set $\Pi_i(x)$ of projections of x onto S_i
  - is closed and bounded
  - may possibly empty in infinite dimension
  - may contain more than one point

- Almost uniqueness of projections: the set of points which have more then one projection onto a set is "negligible"

PROJECTION ONTO THE FOURIER MODULUS SET

$$S_i = \left\{ x \in L^2 \mid |\widehat{x}| 1_A = \mathfrak{m} 1_A \right\}$$

- S_i is neither convex nor weakly closed.

- Every signal $x \in L^2$ has at least one projection onto S_i which is defined by (see p. 8)

  $$\widehat{p_i} = \widehat{x} 1_{\complement A} + \mathfrak{m} \exp(i \angle \widehat{x}) 1_{A \setminus B} + \mathfrak{m} \exp(i\varphi) 1_B,$$

  where $B \subset A$ is such that $\widehat{x} 1_B = 0$ a.e. and $\varphi \colon B \to [0, 2\pi[$ is any (measurable) function.

- The projection is unique if $\mathfrak{m} 1_B = 0$ a.e., in particular if $\widehat{x} 1_A \neq 0$ a.e.

Remark: If $A = \mathbb{R}$ and x has compact support, then $\widehat{x} \neq 0$ a.e. (the Fourier zeros of a compactly supported function are isolated). Hence, in the Gerchberg-Saxton algorithm, projections are always unique since support truncation precedes the projection onto S_i.

METHOD OF SUCCESSIVE PROJECTIONS (MOSP)

- MOSP consists in projecting sequentially an initial estimate onto the sets in a cyclic manner.

- It is described by the recursion

$$(\forall n \in \mathbb{N}) \quad x_{n+1} \in \Pi_{n \pmod{m}+1}(x_n),$$

  where Π_i is the projector onto S_i.

- In the convex case, MOSP coincides with POCS (projection onto convex sets).

- MOSP converges locally (PLC & Trussell, 1990)

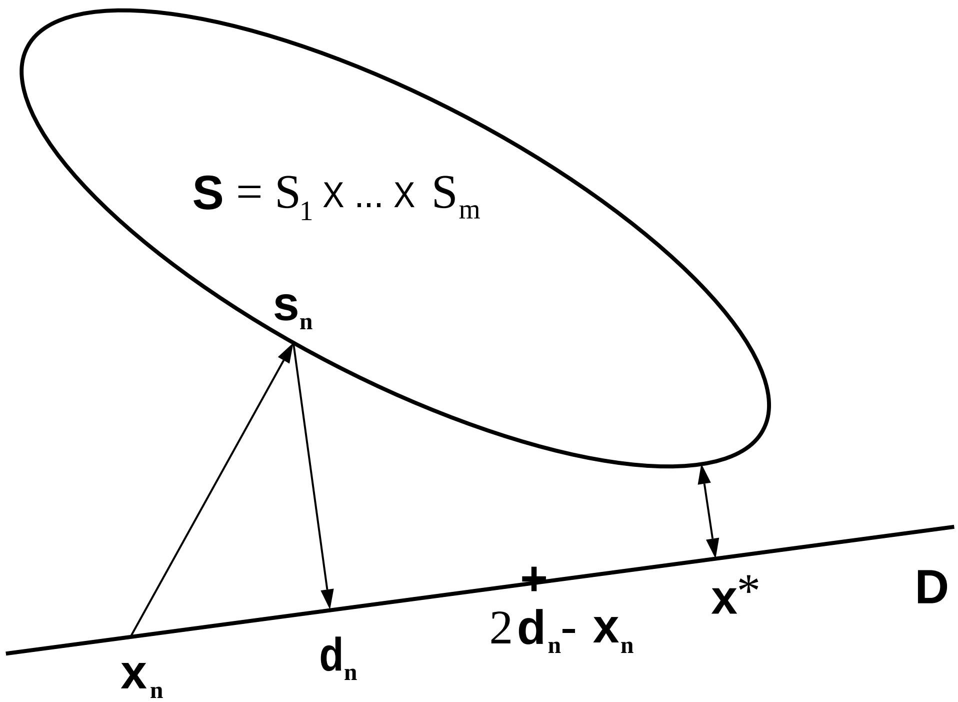
PARALLEL PROJECTIONS METHOD (PPM)

- Apply MOSP with the 2 sets of the product space $\mathbf{H}$ (p. 18).

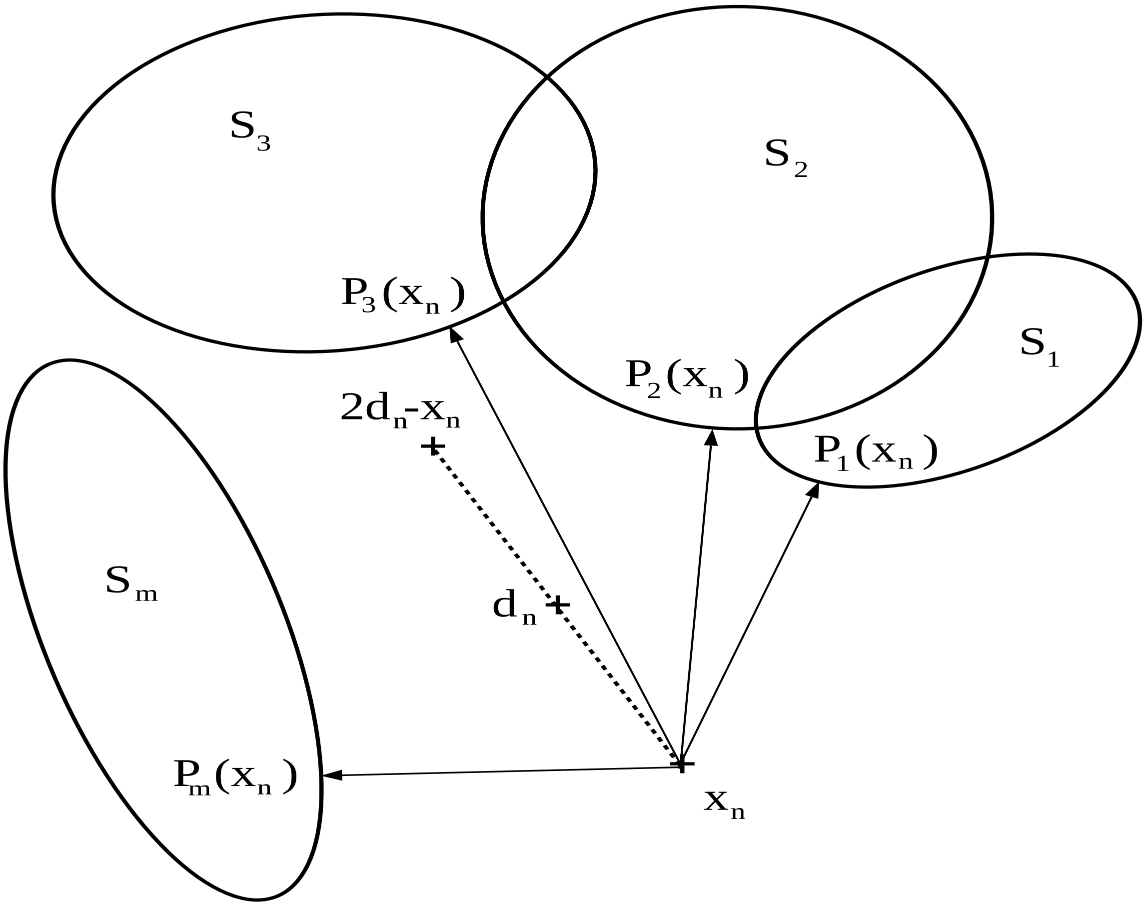
- The algorithm is therefore defined by the set-valued recursion

$$x_{n+1} \in \left\{ \sum_{i \in I} \omega_i p_i \mid (\forall i \in I) \;\; p_i \in \Pi_i(x_n) \right\}.$$

PPM ALGORITHM IN THE PRODUCT SPACE (CONVEX CASE)



PPM ALGORITHM IN THE ORIGINAL SPACE (CONVEX CASE)



CONVERGENCE OF PPM: THE CONVEX CASE

- Assume that, in addition to the above conditions, the sets are convex.
- The recursion becomes

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \sum_{i \in I} \omega_i P_i(x_n).$$

- Define

$$\begin{aligned} \Phi\colon \quad \mathcal{H} &\to [0, +\infty[\\ x &\mapsto \frac{1}{2} \sum_{i \in I} \omega_i d(a, S_i)^2 \end{aligned}$$

- Let G be the set of global minimizers of Φ (the approximate feasible solutions when the set theoretic formulation is inconsistent).
- If $S \neq \emptyset$, then $G = S = \{x \in \mathcal{H} \mid \Phi(x) = 0\}$.
- Suppose that one of the sets is bounded. Then $G \neq \emptyset$ and every sequence $(x_n)_{n \geq 0}$ generated by PPM converges weakly to a point in G.

CONVERGENCE OF PPM: THE GENERAL CASE

- Assume that all the S_i's are boundedly weakly compact and that one of them is weakly compact.
- PPM $x_{n+1} \in \Gamma(x_n)$ where $\Gamma = \sum_{i \in I} \omega_i \Pi_i$.
- Definitions:
  - L is the set of local minimizers of Φ
  - $F = \{x \in \mathcal{H} \mid \{x\} \subset \Gamma(x)\}$ the fixed points set of Γ
  - $T = \{x \in \mathcal{H} \mid \{x\} = \Gamma(x)\}$ the stationary points of Γ
  - C is the set of all cluster points of all $(x_n)_{n \geq 0}$ of Γ.
- Result: $S \subset G \subset L \subset T \subset F = C \neq \emptyset$.
- In practice, this result can be strengthened by noting that $(F \setminus S)^{\complement}$ is dense in $\mathcal{H}$. Thus,
  - Almost every fixed point is a stationary point.
  - Any cluster point is a local minimum, and constitutes a local approximate solution to the feasibility problem.
  - If $S \neq \emptyset$ and x_0 lies in a suitable region of attraction, every cluster point of an orbit $(x_n)_{n \geq 0}$ is feasible.

BREGMAN PROJECTIONS

- In the above algorithms, the difference between two signals x and y is measured by $D(x,y) = \|x-y\|^2/2$.
- Let $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ be a convex, differentiable function.
- The Bregman "distance" between x and y is

$$D(x,y) = f(x) - f(y) - \langle x-y \mid \nabla f(y)\rangle.$$

- For $f = \|\cdot\|^2/2$, one recovers $D(x,y) = \|x-y\|^2/2$.
- In $\mathbb{R}^N$, if f is Shannon's negentropy,

$$f\colon \left(x^{(i)}\right)_{1\leq i\leq N} \mapsto \begin{cases} \sum_{i=1}^N x^{(i)} \ln x^{(i)} & \text{if } x \geq 0 \\ +\infty & \text{otherwise,} \end{cases}$$

one obtains the Kullback-Leibler divergence between $x \geq 0$ and $y > 0$,

$$D(x,y) = \sum_{i=1}^N x^{(i)} \ln\left(x^{(i)}/y^{(i)}\right) - \sum_{i=1}^N x^{(i)} + \sum_{i=1}^N y^{(i)}.$$

BREGMAN PROJECTIONS (cont'd)

- Potentially, the standard projections in the previous algorithms can be replaced by Bregman projections: p_i is a Bregman projection of x onto S_i if

$$D(p_i, x) = \inf_{y \in S_i} D(y, x)$$

- Bregman projections have not yet been used in nonconvex problems. In some convex medical imaging problems they have been reported to be better than standard methods.

- An orbit generated by Bregman projections follows a very different path than one generated via classical projections.